Closed Orbit

Kai Hock

Cockcroft Institute / Liverpool University

June 22, 2010

If all the magnets in a storage ring are perfectly aligned, and there is no radiation loss through magnets or energy gain in cavities, an electron would be able to travel along the design path. Then the displacement vector, $(x, x', y, y', z, \delta, 1)$, or trajectory as it is called in [1], would just be $(0, 0, 0, 0, 0, 0, 1)$.

In reality, misalignments of magnets would result in changes in magnetic fields. This has the effect of forces, or kicks, that would disturb the electron from the design path. Radiation losses or energy gain would cause an electron to bend away from the design path in bending magnets. As a result, the electron would not travel along the design path. The result of the imperfections is that there would effectively be a new "design" path. This new path is called the closed orbit.

The closed orbit can be determined in terms of a displacement from the original design path. The position along the original path is denoted by s. At each s, the electron would be displaced to a corresponding point in the close orbit. Let this displacement be

$$V(s) = \begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \delta \\ 1 \end{pmatrix} \tag{1}$$

Since this is a closed orbit, the electron must return to the same displacement after travelling one round around the ring. Let $W(s)$ be the transformation matrix for one revolution. When multiplied on the displacement at position s, it will give the new displacement after one turn. If the initial displacement is on the closed orbit, then so is the final displacement. So if $V(s)$ is on the closed orbit, it would obey the following equation:

$$W(s)V(s) = V(s) \tag{2}$$

$W(s)$ is obtained by multiplying together the transfer matrices of all of the elements around the ring. It would depend on the initial position s, since the

order of multiplying the elements would be different for a different position. The equation can then be solved for V(s).

In [1], $V(s)$ is described as an eigenvector of the above equation, with an eigenvalue of 1. It also mentions that it is easy to prove that one of the eigenvalue of $W(s)$ is 1. To understand why this is the case, we could look at another way of solving the problem - by applying Gaussian elimination.

First, we note that the last row of $W(s)$ is all zero except for the diagonal term. To see this, look at the transfer matrices of a few elements:

Quadrupole magnet:

$$\begin{pmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ z_1 \\ \delta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ m_{21} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & m_{43} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \tag{3}$$

Horizontal bending magnet:

$$\begin{pmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ z_1 \\ \delta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & m_{26} & m_{27} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ m_{51} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & m_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \tag{4}$$

rf cavity:

$$\begin{pmatrix} x_1 \\ x_1' \\ y_1 \\ y_1' \\ z_1 \\ \delta_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{65} & 1 & m_{67} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_0' \\ y_0 \\ y_0' \\ z_0 \\ \delta_0 \\ 1 \end{pmatrix} \tag{5}$$

It is clear from these examples that the last row is all zero except for the final element, which is 1. When any two matrices with this property are multiplied, the product also has the same property. This can be proved by actually following the steps for calculating the last row of a product.

So, in general, the expanded form of eq. (2) is

$$m_{11}x + m_{12}x' + m_{13}y + m_{14}y' + m_{15}z + m_{16}\delta + m_{17} = x \quad (6)$$

$$m_{21}x + m_{22}x' + m_{23}y + m_{24}y' + m_{25}z + m_{26}\delta + m_{27} = x' \quad (7)$$

$$m_{31}x + m_{32}x' + m_{33}y + m_{34}y' + m_{35}z + m_{36}\delta + m_{37} = y \quad (8)$$

$$m_{41}x + m_{42}x' + m_{43}y + m_{44}y' + m_{45}z + m_{46}\delta + m_{47} = y' \quad (9)$$

$$m_{51}x + m_{52}x' + m_{53}y + m_{54}y' + m_{55}z + m_{56}\delta + m_{57} = z \quad (10)$$

$$m_{61}x + m_{62}x' + m_{63}y + m_{64}y' + m_{65}z + m_{66}\delta + m_{67} = \delta \quad (11)$$

$$1 = 1 \quad (12)$$

There are 6 unknowns $(x, x', y, y', z, \delta)$ and 6 equations. As long as they are all independent, the unique solution can be obtained by Gaussian elimination. This also means that the solution $V(s)$ to eq. (2) exists, and therefore there exists an eigenvalue of 1.

By finding this solution for a series of positions around the ring, the closed orbit may be obtained.

There is one problem. Recall the matrix elements for the sextupole [2]. These are functions of x and y. We need to know the matrix before we can find the x and y in the closed orbit, but we need to know x and y before we can find the matrix. To get around this chicken and egg problem, [1] uses an iterative method:

1. Set x and y to zero at a particular position s.

2. Find the matrix $W(s)$.

3. Find the solution vector $V(s)$.

4. Get x and y from this solution.

5. Find the matrix $W(s)$ again using these new values of x and y.

6. Find the solution $V(s)$.

7. Repeat until the values of x and y converge.

References

[1] Alex Chao, Evaluation of Beam Distribution Parameters in an Electron Storage Ring, SLAC-PUB-2143. `http://www.slac.stanford.edu/pubs/slacpubs/2000/slac-pub-2143.html`

[2] `http://hep.ph.liv.ac.uk/~hock/Damping_Ring/sextupole.pdf`